

# ON SOME CHARACTERIZATIONS OF RULED SURFACE OF A CLOSED TIMELIKE CURVE IN DUAL LORENTZIAN SPACE

Özcan BEKTAŞ \*      Süleyman ŞENYURT \*

## Abstract

In this paper, we investigate the relations between the pitch, the angle of pitch and drall of parallel ruled surface of a closed curve in dual Lorentzian space.

**Keywords:** Timelike dual curve; ruled surface; Lorentzian space; dual numbers.

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## 1 Introduction

Dual numbers were introduced by W.K. Clifford (1849-79) as a tool for his geometrical investigations. After him E.Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [4]. The pitches and the angles of the pitches of the closed ruled surfaces corresponds to the one parameter dual unit spherical curves in space of lines  $IR^3$  were calculated respectively by Hacısalihoğlu [7] and Gursoy [5]. Definition of the parallel ruled surface were presented by Blaschke (translated by Erim [3]). The integral invariants of the parallel ruled surfaces in the 3-dimensional Euclidean space  $E^3$  corresponding to the unit dual spherical parallel curves were calculated by Senyurt [11]. The set  $ID = \{\hat{\lambda} = \lambda + \varepsilon\lambda^* \mid \lambda, \lambda^* \in IR, \varepsilon^2 = 0\}$  is called dual numbers set [2].

On this set product and addition operations are respectively

$$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$$

and

$$(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = \lambda\beta + \varepsilon(\lambda\beta^* + \lambda^*\beta).$$

$ID^3 = \{\vec{A} = \vec{a} + \varepsilon\vec{a}^* \mid \vec{a}, \vec{a}^* \in IR^3\}$  the elements of  $ID^3$  are called dual vectors. On this set addition and scalar product operations are respectively

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\*Ordu University, Faculty of Art and Science, Department of Mathematics, 52750, Perşembe, Ordu, Turkey, ozcanbektas1986@hotmail.com, senyurtsuleyman@hotmail.com.

$$\begin{aligned} \oplus : ID^3 \times ID^3 &\rightarrow ID^3 \\ (\vec{A}, \vec{B}) &\rightarrow \vec{A} \oplus \vec{B} = \vec{a} + \vec{b} + \varepsilon (\vec{a}^* + \vec{b}^*) \end{aligned}$$

$$\begin{aligned} \odot : ID \times ID^3 &\rightarrow ID^3 \\ (\lambda, \vec{A}) &\rightarrow \lambda \odot \vec{A} = (\lambda + \varepsilon \lambda^*) \odot (\vec{a} + \varepsilon \vec{a}^*) = \lambda \vec{a} + \varepsilon (\lambda \vec{a}^* + \lambda^* \vec{a}) \end{aligned}$$

The set  $(ID^3, \oplus)$  is a module over the ring  $(ID, +, \cdot)$ . ( $ID$ -Modul).

The Lorentzian inner product of dual vectors  $\vec{A}, \vec{B} \in ID^3$  is defined by

$$\langle \vec{A}, \vec{B} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon (\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle)$$

with the Lorentzian inner product  $\vec{a} = (a_1, a_2, a_3)$  and  $\vec{b} = (b_1, b_2, b_3) \in IR^3$

$$\langle \vec{a}, \vec{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Therefore,  $ID^3$  with the Lorentzian inner product  $\langle \vec{A}, \vec{B} \rangle$  is called 3-dimensional dual Lorentzian space and denoted by of  $ID_1^3 = \{ \vec{A} = \vec{a} + \varepsilon \vec{a}^* \mid \vec{a}, \vec{a}^* \in IR_1^3 \}$  [14].

A dual vector  $\vec{A} = \vec{a} + \varepsilon \vec{a}^* \in ID_1^3$  is called A dual space-like vector if  $\langle \vec{A}, \vec{A} \rangle > 0$  or  $\vec{A} = 0$ , A dual time-like vector if  $\langle \vec{A}, \vec{A} \rangle < 0$ , A dual null(light-like) vector if  $\langle \vec{A}, \vec{A} \rangle = 0$  for  $\vec{A} \neq 0$ . For  $\vec{A} \neq 0$ , the norm  $\|\vec{A}\|$  of  $\vec{A} = \vec{a} + \varepsilon \vec{a}^* \in ID^3$  is defined by

$$\|\vec{A}\| = \sqrt{|\langle \vec{A}, \vec{A} \rangle|} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}, \quad \|\vec{a}\| \neq 0.$$

The dual Lorentzian cross-product of  $\vec{A}, \vec{B} \in ID^3$  is defined as

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon (\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b})$$

with the Lorentzian cross-product  $\vec{a}, \vec{b} \in IR^3$

$$\vec{a} \wedge \vec{b} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1) [14].$$

**Theorem(E. Study):** The oriented lines in are in one to one correspondence with the points of the dual unit sphere where ID-Modul, see [9].

Dual number  $\Phi = \varphi + \varepsilon \varphi^*$  is called dual angle between  $\vec{A}$  ve  $\vec{B}$  unit dual vectors. In this place

$$\sinh(\varphi + \varepsilon \varphi^*) = \sinh \varphi + \varepsilon \varphi^* \cosh \varphi \quad \text{and} \quad \cosh(\varphi + \varepsilon \varphi^*) = \cosh \varphi + \varepsilon \varphi^* \sinh \varphi.$$

## 2 ON SOME CHARACTERIZATIONS OF RULED SURFACE OF A CLOSED TIMELIKE CURVE IN DUAL LORENTZIAN SPACE ( $ID_1^3$ )

$\vec{U} = \vec{U}_1(t)$  ,  $\|\vec{U}(t)\| = 1$  is a differentiable timelike curve in the one parameter dual unit spherical motion  $K/K'$ . The closed ruled surface ( $\vec{U}$ ) corresponds to this curve in  $IR^3$ .

Let the dual orthonormal system of curve  $\vec{U} = \vec{U}_1(t)$  as

$$\vec{U}_1 = \vec{U}(t) \quad , \quad \vec{U}_2 = \frac{\vec{U}'(t)}{\|\vec{U}'(t)\|} \quad , \quad \vec{U}_3 = \vec{U}_1 \wedge \vec{U}_2$$

Let  $\vec{U}(t)$  be a closed timelike curve with curvature  $\kappa = k_1 + \varepsilon k_1^*$  and torsion  $\tau = k_2 + \varepsilon k_2^*$ . Let Frenet frames of  $\vec{U}(t)$  be  $\{\vec{U}_1, \vec{U}_2, \vec{U}_3\}$ . In this trihedron,  $\vec{U}_1$  is timelike vector,  $\vec{U}_2$  and  $\vec{U}_3$  are spacelike vectors. For this vectors, we can write

$$\vec{U}_1 \wedge \vec{U}_2 = -\vec{U}_3 \quad , \quad \vec{U}_2 \wedge \vec{U}_3 = \vec{U}_1 \quad , \quad \vec{U}_3 \wedge \vec{U}_1 = -\vec{U}_2 \quad (2.1)$$

where  $\wedge$  is the Lorentzian cross product, in space  $ID_1^3$ . In this situation, the Frenet formulas are given by

$$\vec{U}_1' = \kappa \vec{U}_2 \quad , \quad \vec{U}_2' = \kappa \vec{U}_1 - \tau \vec{U}_3 \quad , \quad \vec{U}_3' = \tau \vec{U}_2 \quad , [15]. \quad (2.2)$$

If the last equation is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{u}_1' = k_1 \vec{u}_2 \\ \vec{u}_2' = k_1 \vec{u}_1 - k_2 \vec{u}_3 \\ \vec{u}_3' = k_2 \vec{u}_2 \\ \vec{u}_1^{*'} = k_1^* \vec{u}_2 + k_1 \vec{u}_2^* \\ \vec{u}_2^{*'} = k_1^* \vec{u}_1 - k_2^* \vec{u}_3 + k_1 \vec{u}_1^* - k_2 \vec{u}_3^* \\ \vec{u}_3^{*'} = k_2^* \vec{u}_2 + k_2 \vec{u}_2^* \end{cases} \quad (2.3)$$

The Frenet instantaneous rotation vector for the timelike curve is given by

$$\vec{\Psi} = \tau \vec{U}_1 - \kappa \vec{U}_3, [13]. \quad (2.4)$$

In this situation for the Steiner vector of the motion, we may write

$$\vec{D} = \oint \vec{\Psi} \quad (2.5)$$

or

$$\vec{D} = \vec{U}_1 \oint \tau dt - \vec{U}_3 \oint \kappa dt \quad (2.6)$$

The equation (2.6) can be written type of the dual and real part as follow

$$\begin{cases} \vec{d} = \vec{u}_1 \oint k_2 dt - \vec{u}_3 \oint k_1 dt, \\ \vec{d}^* = \vec{u}_1^* \oint k_2 dt + \vec{u}_1 \oint k_2^* dt - \vec{u}_3^* \oint k_1 dt - \vec{u}_3 \oint k_1^* dt \end{cases} \quad (2.7)$$

Now, let is calculate the integral invariants of the closed ruled surfaces respectively. The pitch of the closed surface is obtained as

$$L_{u_1} = \langle \vec{d}, \vec{u}_1^* \rangle + \langle \vec{d}^*, \vec{u}_1 \rangle, \quad (2.8)$$

$$L_{u_1} = - \oint k_2^* dt.$$

For the dual angle of the pitch of the closed surface , we may write

$$\Lambda_{U_1} = - \langle \vec{D}, \vec{U}_1 \rangle,$$

Because of the equation (2.6) we can obtain

$$\Lambda_{U_1} = \oint \tau dt. \quad (2.9)$$

If the equation (2.9) is separated into the dual and real part, we can obtain

$$\lambda_{u_1} = \oint k_2 dt \quad , \quad L_{u_1} = - \oint k_2^* dt \quad (2.10)$$

For the drall of the closed surface , we may write

$$P_{U_1} = \frac{\langle d\vec{u}_1^*, d\vec{u}_1^* \rangle}{\langle d\vec{u}_1, d\vec{u}_1 \rangle}$$

Setting by the values of the statements  $d\vec{u}_1^*$  and  $d\vec{u}_1$  as the equations (2.3) into the last equations, we get

$$P_{U_1} = \frac{k_1^*}{k_1} \quad (2.11)$$

The pitch of the closed surface is obtained as

$$L_{u_2} = \langle \vec{d}, \vec{u}_2^* \rangle + \langle \vec{d}^*, \vec{u}_2 \rangle, \quad (2.12)$$

$$L_{u_2} = 0.$$

For the dual angle of the pitch of the closed surface , we may write

$$\Lambda_{U_2} = - \langle \vec{D}, \vec{U}_2 \rangle, \quad (2.13)$$

$$\Lambda_{U_2} = 0.$$

For the drall of the closed surface , we may write

$$P_{U_2} = \frac{\langle \vec{du}_2, \vec{du}_2^* \rangle}{\langle \vec{du}_2, \vec{du}_2 \rangle}$$

Setting by the values of the statements  $\vec{du}_2$  and  $\vec{du}_2^*$  as the equations (2.3) into the last equations, we get

$$P_{U_2} = \frac{k_2 k_2^* - k_1 k_1^*}{k_2^2 - k_1^2} \quad (2.14)$$

The pitch of the closed surface is obtained as

$$L_{u_3} = \langle \vec{d}, \vec{u}_3^* \rangle + \langle \vec{d}^*, \vec{u}_3 \rangle, \quad (2.15)$$

$$L_{u_3} = - \oint k_1^* dt$$

For the dual angle of the pitch of the closed surface , we may write

$$\Lambda_{U_3} = - \langle \vec{D}, \vec{U}_3 \rangle$$

Because of the equation (2.6) we can obtain

$$\Lambda_{U_3} = \oint \kappa dt \quad (2.16)$$

If the equation (2.16) is separated into the dual and real part, we can obtain

$$\lambda_{u_3} = \oint k_1 dt \quad , \quad L_{u_3} = - \oint k_1^* dt \quad (2.17)$$

For the drall of the closed surface , we may write

$$P_{U_3} = \frac{\langle \vec{du}_3, \vec{du}_3^* \rangle}{\langle \vec{du}_3, \vec{du}_3 \rangle}$$

Setting by the values of the statements  $\vec{du}_3$  and  $\vec{du}_3^*$  as the equations (2.3) into the last equations, we get

$$P_{U_3} = \frac{k_2^*}{k_2}. \quad (2.18)$$

Let  $\Omega(t) = \omega(t) + \varepsilon \omega^*(t)$  be Lorentzian timelike angle of between the instantaneous dual Pfaffion vector  $\vec{\Psi}$  and the vector  $\vec{U}_3$ .

**a)** If the instantaneous dual Pfaffion vector  $\vec{\Psi}$  is spacelike ( $|\kappa| > |\tau|$ )

$$\kappa = \|\vec{\Psi}\| \cosh \Omega \quad , \quad \tau = \|\vec{\Psi}\| \sinh \Omega$$

On the way  $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ , unit vector about the vector  $\vec{\Psi}$  direction is

$$\vec{C} = \sinh \Omega \vec{U}_1 - \cosh \Omega \vec{U}_3 \quad (2.19)$$

If the equation (2.19) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{C} = \sinh \omega \vec{u}_1 - \cosh \omega \vec{u}_3 \\ \vec{C}^* = \sinh \omega \vec{u}_1^* - \cosh \omega \vec{u}_3^* + \omega^* \cosh \omega \vec{u}_1 - \omega^* \sinh \omega \vec{u}_3 \end{cases} \quad (2.20)$$

The pitch of the closed surface is obtained as

$$L_C = \langle \vec{d}, \vec{C}^* \rangle + \langle \vec{d}^*, \vec{C} \rangle$$

$$\begin{aligned} L_C = \cosh \omega \oint k_1^* dt - \sinh \omega \oint k_2^* dt - \\ - \omega^* \left( \cosh \omega \oint k_2 dt - \sinh \omega \oint k_1 dt \right) \end{aligned} \quad (2.21)$$

If we use the equations (2.10) and (2.17) into the equation (2.21) we get

$$L_C = \sinh \omega L_{u_1} - \cosh \omega L_{u_3} - \omega^* (\cosh \omega \lambda_{u_1} - \sinh \omega \lambda_{u_3}) \quad (2.22)$$

For the dual angle of the pitch of the closed ruled surface , we may write

$$\Lambda_C = - \langle \vec{D}, \vec{C} \rangle$$

Because of the equations (2.6) and (2.19) we can obtain

$$\Lambda_C = \sinh \Omega \oint \tau dt - \cosh \Omega \oint \kappa dt \quad (2.23)$$

If we use the equations (2.9) and (2.16) into the last equations, we get

$$\Lambda_C = \sinh \Omega \Lambda_{U_1} - \cosh \Omega \Lambda_{U_3} \quad (2.24)$$

For the drall of the closed surface , we may write

$$P_C = \frac{\langle d\vec{C}, d\vec{C}^* \rangle}{\langle d\vec{C}, d\vec{C} \rangle}$$

$$P_C = \frac{-\omega' \omega^{*'} + (k_1 \sinh \omega - k_2 \cosh \omega) [(k_1^* - k_2 \omega^*) \sinh \omega + (k_1 \omega^* - k_2^*) \cosh \omega]}{(k_1 \sinh \omega - k_2 \cosh \omega)^2 - \omega'^2} \quad (2.25)$$

**b)** If the instantaneous dual Pfaffion vector  $\vec{\Psi}$  is timelike ( $|\kappa| < |\tau|$ )

$$\kappa = \left\| \vec{\Psi} \right\| \sinh \Omega \quad , \quad \tau = \left\| \vec{\Psi} \right\| \cosh \Omega$$

On the way  $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ , unit vector about the vector  $\vec{\Psi}$  direction is

$$\vec{C} = \cosh \Omega \vec{U}_1 - \sinh \Omega \vec{U}_3 \quad (2.26)$$

If the equation (2.26) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{c} = \cosh \omega \vec{u}_1 - \sinh \omega \vec{u}_3 \\ \vec{c}^* = \cosh \omega \vec{u}_1^* - \sinh \omega \vec{u}_3^* + \omega^* \sinh \omega \vec{u}_1 - \omega^* \cosh \omega \vec{u}_3 \end{cases} \quad (2.27)$$

The pitch of the closed surface is obtained as

$$L_C = \langle \vec{d}, \vec{c}^* \rangle + \langle \vec{d}^*, \vec{c} \rangle$$

$$\begin{aligned} L_C = \sinh \omega \oint k_1^* dt - \cosh \omega \oint k_2^* dt - \\ - \omega^* \left( \sinh \omega \oint k_2 dt - \cosh \omega \oint k_1 dt \right) \end{aligned} \quad (2.28)$$

If we use the equations (2.10) and (2.17) into the equation (2.21) we get

$$L_C = \cosh \omega L_{u_1} - \sinh \omega L_{u_3} - \omega^* (\sinh \omega \lambda_{u_1} - \cosh \omega \lambda_{u_3}) \quad (2.29)$$

For the dual angle of the pitch of the closed ruled surface , we may write

$$\Lambda_C = - \langle \vec{D}, \vec{C} \rangle$$

Because of the equations (2.6) and (2.19) we can obtain

$$\Lambda_C = \cosh \Omega \oint \tau dt - \sinh \Omega \oint \kappa dt \quad (2.30)$$

If we use the equations (2.9) and (2.16) into the last equations, we get

$$\Lambda_C = \cosh \Omega \Lambda_{U_1} - \sinh \Omega \Lambda_{U_3} \quad (2.31)$$

For the drall of the closed surface , we may write

$$P_C = \frac{\langle d\vec{c}, d\vec{c}^* \rangle}{\langle d\vec{c}, d\vec{c} \rangle}$$

$$P_C = \frac{\omega' \omega^{*'} + (k_1 \cosh \omega - k_2 \sinh \omega) [(k_1^* - k_2 \omega^*) \cosh \omega + (k_1 \omega^* - k_2^*) \sinh \omega]}{(k_1 \sinh \omega - k_2 \cosh \omega)^2 - \omega'^2} \quad (2.32)$$

**Definition:** The closed ruled surface  $(\vec{U})$  corresponds to the dual timelike curve  $\vec{U}(t)$  which makes the fixed dual angle  $\Phi = \varphi + \varepsilon \varphi^*$  with  $\vec{U}(t)$  and defines by

$$\vec{V} = \cosh \Phi \vec{U}_1 + \sinh \Phi \vec{U}_3 \quad (2.33)$$

This surface ( $\vec{V}$ ) corresponds to dual timelike vector  $\vec{V}$  is called the parallel ruled surface of surface ( $\vec{U}$ ) in dual lorentzian space  $ID_1^3$ .

Let be  $\vec{V}_1 = \vec{V}$ . Differentiating of the vector  $\vec{V}_1$  with respect the parameter and using the equation (2.3) we get

$$\vec{V}_1' = (\kappa \cosh \Phi + \tau \sinh \Phi) \vec{U}_2 \quad (2.34)$$

If the norm of the vector denotes by  $P$ , we get

$$\vec{P} = \kappa \cosh \Phi + \tau \sinh \Phi \quad (2.35)$$

Then if is known that

Substituting by the values of the equations (2.34) and (2.35) into , we get

$$\vec{V}_2 = \vec{U}_2 \quad (2.36)$$

Then, fort he vector , we get

$$\vec{V}_3 = -\sinh \Phi \vec{U}_1 - \cosh \Phi \vec{U}_3 \quad (2.37)$$

If the equation (2.33) , (2.36) and (2.37) are written matrix form, we have

$$\begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \end{bmatrix} = \begin{bmatrix} \cosh \Phi & 0 & \sinh \Phi \\ 0 & 1 & 0 \\ -\sinh \Phi & 0 & -\cosh \Phi \end{bmatrix} \cdot \begin{bmatrix} \vec{U}_1 \\ \vec{U}_2 \\ \vec{U}_3 \end{bmatrix} \quad (2.38)$$

or

$$\begin{bmatrix} \vec{U}_1 \\ \vec{U}_2 \\ \vec{U}_3 \end{bmatrix} = \begin{bmatrix} \cosh \Phi & 0 & \sinh \Phi \\ 0 & 1 & 0 \\ -\sinh \Phi & 0 & -\cosh \Phi \end{bmatrix} \cdot \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \end{bmatrix} \quad (2.39)$$

If the equation (2.39) is separated into real and dual parts, we get

$$\begin{cases} \vec{u}_1 = \cosh \varphi \vec{v}_1 + \sinh \varphi \vec{v}_3 \\ \vec{u}_2 = \vec{v}_2 \\ \vec{u}_3 = -\sinh \varphi \vec{v}_1 - \cosh \varphi \vec{v}_3 \\ \vec{u}_1^* = \cosh \varphi \vec{v}_1^* + \sinh \varphi \vec{v}_3^* + \varphi^* (\sinh \varphi \vec{v}_1 + \cosh \varphi \vec{v}_3) \\ \vec{u}_2^* = \vec{v}_2^* \\ \vec{u}_3^* = -\sinh \varphi \vec{v}_1^* - \cosh \varphi \vec{v}_3^* - \varphi^* (\cosh \varphi \vec{v}_1 + \sinh \varphi \vec{v}_3) \end{cases} \quad (2.40)$$

Let be curvature  $P = p + \varepsilon p^*$  and torsion  $Q = q + \varepsilon q^*$  of curve  $\vec{V}(t)$ . Between the vectors  $\vec{V}_1, \vec{V}_2, \vec{V}_3$  and the derivate vectors  $\vec{V}_1', \vec{V}_2', \vec{V}_3'$  there is following relation



$$\begin{cases} \vec{V}_1' = P\vec{V}_2, & \vec{V}_2' = P\vec{V}_1 - Q\vec{V}_3, & \vec{V}_3' = Q\vec{V}_2 \\ P = \sqrt{\langle \vec{V}_1', \vec{V}_1' \rangle}, & Q = \frac{\det(\vec{V}_1, \vec{V}_1', \vec{V}_1'')}{\langle \vec{V}_1, \vec{V}_1 \rangle}. \end{cases} \quad [15]. \quad (2.41)$$

If the equation (2.41) is separated into the real and dual parts, we can write

$$\begin{cases} \vec{v}_1' = p\vec{v}_2, & \vec{v}_2' = p\vec{v}_1 - q\vec{v}_3, & \vec{v}_3' = q\vec{v}_2 \\ \vec{v}_1'^* = p\vec{v}_2^* + p^*\vec{v}_2, \\ \vec{v}_2'^* = p\vec{v}_1^* + p^*\vec{v}_1 - q\vec{v}_3^* - q^*\vec{v}_3, \\ \vec{v}_3'^* = q\vec{v}_2^* + q^*\vec{v}_2 \end{cases} \quad (2.42)$$

Now, we can calculate the value of  $Q$  relative to  $\kappa$  and  $\tau$ . Derivative the equation (2.34) with respect to the parameter  $\Phi$  and making the necessary operations we may write

$$\begin{aligned} \vec{V}_1'' &= (\kappa^2 \cosh \Phi + \kappa \tau \sinh \Phi) \vec{U}_1 + \\ &+ (\kappa \cosh \Phi + \tau \sinh \Phi) \vec{U}_2 + (-\kappa \tau \cosh \Phi - \tau^2 \sinh \Phi) \vec{U}_3 \end{aligned} \quad (2.43)$$

Substituting by the equations (2.33), (2.34) and (2.43) into the equation (2.41), we get

$$Q = -\kappa \sinh \Phi - \tau \cosh \Phi \quad (2.44)$$

The equations (2.35) and (2.44) are separated into the dual and real parts, we have

$$\begin{cases} p = k_1 \cosh \varphi + k_2 \sinh \varphi \\ p^* = k_1^* \cosh \varphi + k_2^* \sinh \varphi + \varphi^* (k_1 \sinh \varphi + k_2 \cosh \varphi) \\ q = -k_1 \sinh \varphi - k_2 \cosh \varphi \\ q^* = -k_1^* \sinh \varphi - k_2^* \cosh \varphi - \varphi^* (k_1 \cosh \varphi + k_2 \sinh \varphi) \end{cases} \quad (2.45)$$

In the dual unit spherical motion  $K/K'$ , the dual orthonormal system  $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$  each time  $t$  is make a dual rotation motion around the instantaneous dual Pfaffion vector. This vector is determined the following equation

$$\vec{\Psi} = Q\vec{V}_1 - P\vec{V}_3, [13] \quad (2.46)$$

For the Steiner vector of the motion, we can write

$$\vec{D} = \oint \vec{\Psi} \quad (2.47)$$

or

$$\vec{D} = \vec{V}_1 \oint Q dt - \vec{V}_3 \oint P dt \quad (2.48)$$

Setting by the values of the vectors  $\vec{U}_1$  and  $\vec{U}_3$  as the equations (2.40) into the equations (2.4), we get

$$\vec{\Psi} = -Q\vec{V}_1 + P\vec{V}_3$$

In this case, we consider the last equation and equation (2.46), we can write

$$\vec{\Psi} = -\vec{\overline{\Psi}} \quad (2.49)$$

Because of the equations  $\vec{D} = \oint \vec{\Psi}$  and  $\vec{\overline{D}} = \oint \vec{\overline{\Psi}}$  we can obtain  $\vec{D} = -\vec{\overline{D}}$ . Then, for the dual Steiner vector of the motion, we may write

$$\vec{D} = -\vec{V}_1 \oint Q dt + \vec{V}_3 \oint P dt \quad (2.50)$$

It is separated into the dual and real part as

$$\begin{cases} \vec{d} = -\vec{v}_1 \oint q dt + \vec{v}_3 \oint p dt, \\ \vec{d}^* = -\vec{v}_1^* \oint q^* dt - \vec{v}_1^* \oint q dt + \vec{v}_3 \oint p^* dt + \vec{v}_3^* \oint p dt \end{cases} \quad (2.51)$$

Now, let us calculate the integral invariants of the closed ruled surfaces respectively. The pitch of the closed surface is obtained as

$$\begin{aligned} L_{V_1} &= \langle \vec{d}, \vec{v}_1^* \rangle + \langle \vec{d}^*, \vec{v}_1 \rangle, \\ L_{V_1} &= \oint q^* dt. \end{aligned} \quad (2.52)$$

Substituting by the value into the equation (2.52)

$$\begin{aligned} L_{V_1} &= -\sinh \varphi \oint k_1^* dt - \cosh \varphi \oint k_2^* dt - \\ &\quad - \varphi^* \left( \cosh \varphi \oint k_1 dt + \sinh \varphi \oint k_2 dt \right). \end{aligned} \quad (2.53)$$

or

$$L_{V_1} = \cosh \varphi L_{u_1} + \sinh \varphi L_{u_3} - \varphi^* (\sinh \varphi \lambda_{u_1} + \cosh \varphi \lambda_{u_3}). \quad (2.54)$$

For the dual angle of the pitch of the closed ruled surface, we may write

$$\Lambda_{V_1} = -\langle \vec{D}, \vec{V}_1 \rangle$$

Because of the equation (2.50) we can obtain

$$\Lambda_{V_1} = -\oint Q dt. \quad (2.55)$$

Substituting by the equation (2.44) into the last equation, we get

$$\Lambda_{V_1} = \sinh \Phi \oint \kappa dt + \cosh \Phi \oint \tau dt$$

or

$$\Lambda_{V_1} = \cosh \Phi \wedge_{U_1} + \sinh \Phi \wedge_{U_3} \quad (2.56)$$

If the equation (2.56) is separated into the dual and real part, we can obtain

$$\begin{cases} \lambda_{v_1} = \cosh \varphi \lambda_{u_1} + \sinh \varphi \lambda_{u_3} \\ L_{v_1} = \cosh \varphi L_{u_1} + \sinh \varphi L_{u_3} - \varphi^* (\sinh \varphi \lambda_{u_1} + \cosh \varphi \lambda_{u_3}) \end{cases} \quad (2.57)$$

For the drall of the closed surface , we may write

$$P_{V_1} = \frac{\langle d\vec{v}_1, d\vec{v}_1^* \rangle}{\langle d\vec{v}_1, d\vec{v}_1 \rangle}$$

Setting by the values of the statements  $d\vec{v}_1$  and  $d\vec{v}_1^*$  as the equations (2.42) into the last equations, we get

$$P_{V_1} = \frac{p^*}{p} \quad (2.58)$$

Setting by the values of  $p$  and  $p^*$  as the equations (2.45) into the last equations, we get

$$P_{V_1} = \frac{k_1^* \cosh \varphi + k_2^* \sinh \varphi}{k_1 \cosh \varphi + k_2 \sinh \varphi} + \varphi^* \frac{k_1 \sinh \varphi + k_2 \cosh \varphi}{k_1 \cosh \varphi + k_2 \sinh \varphi} \quad (2.59)$$

**Theorem 2.1:** Let  $(V_1)$  be the parallel surface of the surface  $(U_1)$ . The pitch, drall and the dual of the pitch of the ruled surface  $(V_1)$  are

$$1-) L_{V_1} = \oint q^* dt \quad 2-) \Lambda_{V_1} = - \oint Q dt \quad 3-) P_{V_1} = \frac{p^*}{p}.$$

**Corollary 2.1:** Let  $(V_1)$  be the parallel surface of the surface  $(U_1)$ . The pitch and the dual of the pitch of the ruled surface  $(V_1)$  related to the invariants of the surface  $(U_1)$  are written as follow

$$\begin{aligned} 1-) L_{V_1} &= \cosh \varphi L_{u_1} + \sinh \varphi L_{u_3} - \varphi^* (\sinh \varphi \lambda_{u_1} + \cosh \varphi \lambda_{u_3}) \\ 2-) \Lambda_{V_1} &= \cosh \Phi \wedge U_1 + \sinh \Phi \wedge U_3 \end{aligned}$$

The pitch of the closed surface  $(V_2)$  is obtained as

$$\begin{aligned} L_{V_2} &= \langle \vec{d}, \vec{v}_2^* \rangle + \langle \vec{d}^*, \vec{v}_2 \rangle \\ L_{V_2} &= 0 \end{aligned} \quad (2.60)$$

For the dual angle of the pitch of the closed ruled surface  $(V_2)$ , we may write

$$\Lambda_{V_2} = - \langle \vec{D}, \vec{V}_2 \rangle$$

Because of the equation (2.32) we can obtain

$$\Lambda_{V_2} = 0 \quad (2.61)$$

For the drall of the closed surface  $(V_2)$ , we may write

$$P_{V_2} = \frac{\langle d\vec{v}_2, d\vec{v}_2^* \rangle}{\langle d\vec{v}_2, d\vec{v}_2 \rangle}$$

Setting by the values of the statements  $d\vec{v}_2$  and  $d\vec{v}_2^*$  as the equations (2.24) into the last equations, we get

$$P_{V_2} = \frac{qq^* - pp^*}{q^2 - p^2} \quad (2.62)$$

Setting by the values of  $p, p^*, q$  and  $q^*$  as the equations (2.45) into the last equations, we get

$$P_{V_2} = \frac{k_2k_2^* - k_1k_1^*}{k_2^2 - k_1^2} \quad (2.63)$$

**Theorem 2.2:** Let  $(V_1)$  be the parallel surface of the surface  $(U_1)$ . The pitch, drall and the dual of the pitch of the ruled surface  $(V_2)$  are

$$1-)L_{V_2} = 0 \quad 2-)\Lambda_{V_2} = 0 \quad 3-)P_{V_2} = \frac{qq^* - pp^*}{q^2 - p^2}$$

The pitch of the closed surface  $(V_3)$  is obtained as

$$L_{V_3} = \langle \vec{d}, \vec{v}_3^* \rangle + \langle \vec{d}^*, \vec{v}_3 \rangle, \quad (2.64)$$

$$L_{V_3} = \oint p^* dt$$

Substituting by the value into the equation (2.64)

$$L_{V_3} = \cosh \varphi \oint k_1^* dt + \sinh \varphi \oint k_2^* dt + \varphi^* (\sinh \varphi \oint k_1 dt + \cosh \varphi \oint k_2 dt) \quad (2.65)$$

or

$$L_{V_3} = -\sinh \varphi L_{u_1} - \cosh \varphi L_{u_3} + \varphi^* (\cosh \varphi \lambda_{u_1} + \sinh \varphi \lambda_{u_3}) \quad (2.66)$$

For the dual angle of the pitch of the closed ruled surface, we may write

$$\Lambda_{V_3} = -\langle \vec{D}, \vec{V}_3 \rangle$$

Because of the equation (2.50) we can obtain

$$\Lambda_{V_3} = -\oint P dt. \quad (2.67)$$

Substituting by the equation (2.35) into the last equation, we get

$$\Lambda_{V_3} = -\cosh \Phi \oint \kappa dt - \sinh \Phi \oint \tau dt$$

or

$$\Lambda_{V_3} = -\sinh \Phi \wedge_{U_1} - \cosh \Phi \wedge_{U_3} \quad (2.68)$$

If the equation (2.68) is separated into the dual and real part, we can obtain

$$\begin{cases} \lambda_{v_3} = -\sinh \varphi \lambda_{u_1} - \cosh \varphi \lambda_{u_3} \\ L_{v_3} = -\sinh \varphi L_{u_1} - \cosh \varphi L_{u_3} + \varphi^* (\cosh \varphi \lambda_{u_1} + \sinh \varphi \lambda_{u_3}) \end{cases} \quad (2.69)$$

For the drall of the closed surface , we may write

$$P_{V_3} = \frac{\langle d\vec{v}_3, d\vec{v}_3^* \rangle}{\langle d\vec{v}_3, d\vec{v}_3 \rangle}$$

Setting by the values of the statements  $d\vec{v}_3$  and  $d\vec{v}_3^*$  as the equations (2.42) into the last equations, we get

$$P_{V_3} = \frac{q^*}{q} \quad (2.70)$$

Setting by the values of  $q$  and  $q^*$  as the equations (2.45) into the last equations, we get

$$P_{V_3} = \frac{-k_1^* \sinh \varphi - k_2^* \cosh \varphi}{-k_1 \sinh \varphi - k_2 \cosh \varphi} - \varphi^* \left( \frac{k_1 \cosh \varphi + k_2 \sinh \varphi}{-k_1 \sinh \varphi - k_2 \cosh \varphi} \right) \quad (2.71)$$

**Theorem 2.3:** Let  $(V_1)$  be the parallel surface of the surface  $(U_1)$ . The pitch , drall and the dual of the pitch of the ruled surface  $(V_3)$  are

$$1-) L_{V_3} = \oint p^* dt \quad 2-) \Lambda_{V_3} = - \oint P dt \quad 3-) P_{V_3} = \frac{q^*}{q}$$

**Corollary 2.2:** Let  $(V_1)$  be the parallel surface of the surface  $(U_1)$ . The pitch and the dual of the pitch of the ruled surface  $(V_3)$  related to the invariants of the surface  $(U_1)$  are written as follow

$$\begin{aligned} 1-) L_{V_3} &= -\sinh \varphi L_{u_1} - \cosh \varphi L_{u_3} + \varphi^* (\cosh \varphi \lambda_{u_1} + \sinh \varphi \lambda_{u_3}) \\ 2-) \Lambda_{V_3} &= -\sinh \Phi \wedge_{U_1} - \cosh \Phi \wedge_{U_3} \end{aligned}$$

Let  $\Theta(t) = \theta(t) + \varepsilon \theta^*(t)$  be Lorentzian timelike angle of between the instantaneous dual Pfaffion vector  $\vec{\Psi}$  and the vector  $\vec{V}_3$ .

a) If the instantaneous dual Pfaffion vector  $\vec{\Psi}$  is spacelike ( $|P| > |Q|$ )

$$P = \left\| \vec{\Psi} \right\| \cosh \Theta, \quad Q = \left\| \vec{\Psi} \right\| \sinh \Theta$$

On the way  $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ , unit vector about the vector  $\vec{\Psi}$  direction is

$$\vec{C} = \sinh \Theta \vec{V}_1 - \cosh \Theta \vec{V}_3 \quad (2.72)$$

Setting by the values of the vectors  $\vec{V}_1$  and  $\vec{V}_3$  as the equations (2.38) into the equation (2.72), we get

$$\vec{C} = (\sinh \Theta \cosh \Phi + \cosh \Theta \sinh \Phi) \vec{U}_1 + (\cosh \Theta \cosh \Phi + \sinh \Theta \sinh \Phi) \vec{U}_3$$

$$\vec{\overline{C}} = \sinh(\Theta + \Phi) \vec{U}_1 + \cosh(\Theta + \Phi) \vec{U}_3 \quad (2.73)$$

If the equation (2.72) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{\overline{c}} = \sinh \theta \vec{v}_1 - \cosh \theta \vec{v}_3 \\ \vec{\overline{c}}^* = \sinh \theta \vec{v}_1^* - \cosh \theta \vec{v}_3^* + \theta^* \cosh \theta \vec{v}_1 - \theta^* \sinh \theta \vec{v}_3 \end{cases} \quad (2.74)$$

The pitch of the closed surface is obtained as

$$L_{\overline{C}} = \langle \vec{d}, \vec{\overline{c}}^* \rangle + \langle \vec{d}^*, \vec{\overline{c}} \rangle$$

Setting by the values of the statements  $\vec{d}$  and  $\vec{d}^*$  as the equations (2.51) into the last equations and if we do the necessary operations, we get

$$L_{\overline{C}} = -\cosh \theta \oint p^* dt + \sinh \theta \oint q^* dt + \theta^* \left( \cosh \theta \oint q dt - \sinh \theta \oint p dt \right) \quad (2.75)$$

If we use the equations (2.52), (2.64) into the equation (2.75) we get

$$L_{\overline{C}} = \sinh \theta L_{V_1} - \cosh \theta L_{V_3} + \theta^* (-\cosh \theta \lambda_{V_1} + \sinh \theta \lambda_{V_3}) \quad (2.76)$$

If we use the equations (2.57) and (2.69) into the equation (2.76) and necessary operations have been done, we get

$$\begin{aligned} L_{\overline{C}} = & \sinh(\theta + \varphi) L_{U_1} + \cosh(\theta + \varphi) L_{U_3} - \\ & - (\varphi^* + \theta^*) (\cosh(\theta + \varphi) \lambda_{U_1} + \sinh(\theta + \varphi) \lambda_{U_3}) \end{aligned} \quad (2.77)$$

For the dual angle of the pitch of the closed ruled surface, we may write

$$\Lambda_{\overline{C}} = -\left\langle \vec{D}, \vec{\overline{C}} \right\rangle$$

Because of the equations (2.21) and (2.71) we can obtain

$$\Lambda_{\overline{C}} = -\sinh \Theta \oint Q dt + \cosh \Theta \oint P dt \quad (2.78)$$

If we use the equations (2.55) and (2.67) into the last equations, we get

$$\Lambda_{\overline{C}} = \sinh \Theta \Lambda_{V_1} - \cosh \Theta \Lambda_{V_3} \quad (2.79)$$

If we use the equations (2.56) and (2.68) into the equations (2.79), we get

$$\Lambda_{\overline{C}} = \sinh(\Theta + \Phi) \Lambda_{U_1} + \cosh(\Theta + \Phi) \Lambda_{U_3} \quad (2.80)$$

For the drall of the closed surface , we may write

$$P_{\overline{C}} = \frac{\langle d\vec{c}, d\vec{c}^* \rangle}{\langle d\vec{c}, d\vec{c} \rangle}$$

$$P_{\overline{C}} = \frac{-\theta'\theta^{*'} + (p \sinh \theta - q \cosh \theta) [(p^* - q\theta^*) \sinh \theta + (p\theta^* - q^*) \cosh \theta]}{(p \sinh \theta - q \cosh \theta)^2 - \theta'^2} \quad (2.81)$$

b) If the instantaneous dual Pfaffion vector  $\vec{\Psi}$  is timelike ( $|P| < |Q|$ )

$$P = \left\| \vec{\Psi} \right\| \sinh \Theta \quad , \quad Q = \left\| \vec{\Psi} \right\| \cosh \Theta$$

On the way  $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ , unit vector about the vector  $\vec{\Psi}$  direction is

$$\vec{C} = \cosh \Theta \vec{V}_1 - \sinh \Theta \vec{V}_3 \quad (2.82)$$

Setting by the values of the vectors  $\vec{V}_1$  and  $\vec{V}_3$  as the equations (2.38) into the equation (2.82), we get

$$\vec{C} = (\cosh \Theta \cosh \Phi + \sinh \Theta \sinh \Phi) \vec{U}_1 + (\sinh \Theta \cosh \Phi + \cosh \Theta \sinh \Phi) \vec{U}_3$$

$$\vec{C} = \cosh (\Theta + \Phi) \vec{U}_1 + \sinh (\Theta + \Phi) \vec{U}_3 \quad (2.83)$$

If the equation (2.82) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{c} = \cosh \theta \vec{v}_1 - \sinh \theta \vec{v}_3 \\ \vec{c}^* = \cosh \theta \vec{v}_1^* - \sinh \theta \vec{v}_3^* + \theta^* \sinh \theta \vec{v}_1 - \theta^* \cosh \theta \vec{v}_3 \end{cases} \quad (2.84)$$

The pitch of the closed surface is obtained as

$$L_{\overline{C}} = \langle \vec{d}, \vec{c}^* \rangle + \langle \vec{d}^*, \vec{c} \rangle$$

Setting by the values of the statements  $\vec{d}$  and  $\vec{d}^*$  as the equations (2.51) into the last equations and if we do the necessary operations , we get

$$L_{\overline{C}} = -\sinh \theta \oint p^* dt + \cosh \theta \oint q^* dt + \theta^* \left( \sinh \theta \oint q dt - \cosh \theta \oint p dt \right) \quad (2.85)$$

or

$$L_{\overline{C}} = \cosh \theta L_{V_1} - \sinh \theta L_{V_3} + \theta^* (-\sinh \theta \lambda_{V_1} + \cosh \theta \lambda_{V_3}) \quad (2.86)$$

If we use the equations (2.57) and (2.69) into the equation (2.86) and necessary operations have been done, we get

$$L_{\overline{C}} = \cosh(\theta + \varphi)L_{U_1} + \sinh(\theta + \varphi)L_{U_3} - (\varphi' * + \theta^*)(\sinh(\theta + \varphi)\lambda_{U_1} + \cosh(\theta + \varphi)\lambda_{U_3}) \quad (2.87)$$

For the dual angle of the pitch of the closed ruled surface , we may write

$$\Lambda_{\overline{C}} = - \left\langle \overrightarrow{D}, \overrightarrow{C} \right\rangle$$

Because of the equations (2.50) and (2.82) we can obtain

$$\Lambda_{\overline{C}} = - \cosh \Theta \oint Q dt + \sinh \Theta \oint P dt \quad (2.88)$$

If we use the equations (2.55) and (2.67) into the last equations, we get

$$\Lambda_{\overline{C}} = \cosh \Theta \Lambda_{V_1} - \sinh \Theta \Lambda_{V_3} \quad (2.89)$$

If we use the equations (2.57) and (2.68) into the equations (2.89), we get

$$\Lambda_{\overline{C}} = \cosh(\Theta + \Phi) \Lambda_{U_1} + \sinh(\Theta + \Phi) \Lambda_{U_3} \quad (2.90)$$

For the drall of the closed surface , we may write

$$P_{\overline{C}} = \frac{\left\langle d\overrightarrow{c}, d\overrightarrow{c}^* \right\rangle}{\left\langle d\overrightarrow{c}, d\overrightarrow{c} \right\rangle}$$

$$P_{\overline{C}} = \frac{\theta' \theta^{*'} + (p \cosh \theta - q \sinh \theta) [(p \theta^* - q^*) \sinh \theta + (p^* - q \theta^*) \cosh \theta]}{(p \cosh \theta - q \sinh \theta)^2 + \theta'^2} \quad (2.91)$$

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